



Motivation for Hodge cycles

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Abstract

Given two smooth projective varieties X and Y , X is defined to motivate Y if the motive of Y is contained in the tensor category generated by X . Some techniques are given for checking this condition. It is shown that in a number of cases moduli spaces of sheaves over curves or surfaces are motivated by the underlying curve or surface. This is used to check the Hodge and related conjectures for some of these examples.
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Given two smooth projective varieties X and Y over a field, we say that X *motivates* Y or that Y is *motivated by* X if the motive of Y is contained in the category generated from X by taking sums, summands and products. This notion has appeared implicitly in many places, but it seems useful to isolate it so as to state the following principle (Lemma 4.2): if the Hodge (generalized Hodge, Lefschetz standard...) conjecture holds for X and all its powers, then it holds for any variety motivated by it. For the precise statement we can use homological motives, however, we find it more convenient to use the construction of motives due to André [2] which has the advantage of yielding a (provably) semisimple Abelian category through which cohomology factors.

Given a smooth complex projective variety X , we can take the dimension of the smallest variety that motivates X as a measure of its complexity. This number can be seen to be maximal for general varieties using work of Schoen [41]; however, there are a number of interesting examples, discussed below, where this is small. Varieties motivated by curves are the simplest. For

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such varieties, a weak form of the Hodge conjecture, that Hodge cycles are motivated in André's sense, holds unconditionally. Such cycles are absolutely Hodge in Deligne's sense. Next in line are varieties motivated by curves or surfaces. For these varieties we check that the Lefschetz standard conjecture of Grothendieck holds.

There are a number of natural examples of varieties motivated by curves and surfaces. These include Abelian varieties, uniruled threefolds and unirational fourfolds. These are checked by direct geometric arguments. For Abelian varieties, we first observe that the Jacobian of a curve is motivated by the curve. This generalizes to other moduli spaces. We show that the moduli space of stable parabolic bundles over a curve is motivated by the curve, the Hilbert scheme of points over a surface is motivated by the surface, and likewise for the moduli space of stable vector bundles over an Abelian or K3 surface. For vector bundles over a curve, this result was first proved by del Baño [13]. For the Hilbert scheme of a surface, this goes back to de Cataldo–Migliorini [12]. However, we give uniform and self contained treatments of these cases. Using these results, we check the (generalized) Hodge conjecture for the above spaces in some cases, and the Lefschetz standard conjecture in all cases.

1. Motives

Let k be a field. Let $SPVar_k$ be the category of smooth projective (possibly reducible) varieties over k . The case of primary interest for us is $k = \mathbb{C}$. Given an object of $SPVar_{\mathbb{C}}$, let $H^{\bullet}(X)$ denote singular rational cohomology of X^{an} with its canonical Hodge structure. This takes values in the category PHS of finite direct sums of polarizable rational Hodge structures. The category PHS is a semisimple \mathbb{Q} -linear Abelian category [15, 4.2.3] with tensor products and duals.

We call a full subcategory \mathcal{V} of $SPVar_k$ *admissible* if it contains $spec k, \mathbb{P}_k^1$ and is stable under products, disjoint unions, and connected components. Let $\langle X \rangle$ be the smallest admissible category containing a variety X .

Given an admissible category \mathcal{V} and an object $X \in SPVar_k$, André [2] has constructed a graded \mathbb{Q} -algebra $A_{mot}^{\bullet}(X)$ called the algebra of *motivated cycles* on X modeled on \mathcal{V} . We refer likewise to elements of $A_{mot}^{\bullet}(X_1 \times X_2)$ as motivated correspondences modeled on \mathcal{V} . Fix a Weil cohomology $h^*(X)$, then we can regard $A_{mot}^{\bullet}(X)$ as a subalgebra of $h^{2*}(X)$. A class $\gamma \in A_{mot}^{\bullet}(X)$ if and only if there exists an object $Y \in \mathcal{V}$ and algebraic cycles α, β on $X \times Y$ such that

$$\gamma = p_*(\alpha \cup * \beta),$$

where $p: X \times Y \rightarrow X$ is the projection, and $*$ is the Lefschetz involution with respect to a product polarization [2]. Note that $A_{mot}^{\bullet}(X)$ contains the algebra of algebraic cycles on X , and it would coincide with it assuming Grothendieck's standard conjectures. Motivated cycles forms a good replacement for algebraic cycles in lieu of these conjectures.

By an *intersection theory* on an admissible category \mathcal{V} , we mean a functor R from \mathcal{V}^{op} to commutative rings equipped with pushforwards satisfying the conditions of [33, Section 1]. There are several examples of interest to us:

- (1) $R = K_0$, the Grothendieck group of coherent sheaves.
- (2) The quotient of the rationalized Chow ring $CH(\cdot) \otimes \mathbb{Q}$ by an adequate equivalence relation (e.g. identity, homological, or numerical equivalence).
- (3) The ring $R(\cdot) = A_{mot}(\cdot)$ of motivated cycles modeled on \mathcal{V} as explained above. See [2].

In all but the first case R has a grading.

Given the above data, we can form the category $Cor_R^u(\mathcal{V})$ of (ungraded) R -correspondences in \mathcal{V} with the same objects as \mathcal{V} , and $Hom(X, Y) = R(X \times Y)$. Composition is given by

$$\beta \circ \alpha = p_{XZ*}(p_{XY}^* \alpha \cdot p_{YZ}^* \beta)$$

where $p_{XZ}: X \times Y \times Z \rightarrow X \times Z, \dots$ are the projections. We write Cor_R^u (respectively $Cor_R^u(X)$) etc. for $Cor_R^u(SPVar_k)$ (respectively $Cor_R^u(\langle X \rangle)$) etc. In the cases, where R has a grading, we define the subcategory of graded correspondences $Cor_R \subset Cor_R^u$ by restricting

$$Hom_{Cor_R}(X, Y) = \prod_i R^{\dim X_i}(X_i \times Y)$$

where X_i are the connected components of X . The category of ungraded (respectively graded) R -motives $M_R^u(\mathcal{V})$ ($M_R(\mathcal{V})$) in \mathcal{V} is obtained by taking the pseudo-Abelian completion of $Cor_R^u(\mathcal{V})$ ($Cor_R(\mathcal{V})$) and inverting the so called Lefschetz motive. Alternatively following [29, 42], the objects of $M_R(\mathcal{V})$ can be regarded as triples (X, p, m) , with $X \in Ob \mathcal{V}$, $p \in End(X) = R^{\dim X}(X \times X)$ an idempotent, and m an integer (we will also write this as $(X, p, 0)(m)$). The morphisms are given by

$$Hom((X, p, m), (Y, q, n)) = q \circ [R^{\dim X - m + n}(X \times Y)] \circ p.$$

When $R = CH() \otimes \mathbb{Q}$ (respectively $R = A^*() = im CH^*() \otimes \mathbb{Q} \rightarrow H^{2*}()$), $M_{CH} = M_R$ (respectively $M_{hom} = M_R$) is called the category of Chow (homological) motives. When $R = A_{mot}$ is the ring of motivated cycles, we call $M_A = M_R$ (respectively $M_A(\mathcal{V})$) the category of André motives (modeled on \mathcal{V}).

We have obvious functors $M_{CH} \rightarrow M_{hom} \rightarrow M_A$. These categories are all \mathbb{Q} -linear pseudo-Abelian categories with tensor products and duals (see [42]), and furthermore M_A is semisimple Abelian [2]. We can associate a motive $[X] = (X, id, 0)$ (in any of the previous senses) to a variety $X \in \mathcal{V}$, and this yields a contravariant functor by assigning to $f: X \rightarrow Y$ the transpose of its graph.

Suppose $k = \mathbb{C}$. Then the functors H^\bullet extend to *covariant* functors on M_A as follows. First, recall that a correspondence $\gamma \in Hom_{Cor_A}(X, Y)$ acts on cohomology by $\gamma_*(\alpha) = p_{Y*}(p_X^* \alpha \cup [\gamma])$. Given (X, p, m) define

$$H^i(X, p, m) = p_* H^{i+2m}(X)(m)$$

where (m) represents Tate twist of the canonical Hodge structure. If

$$f \in Hom((X, p, m), (Y, q, n))$$

is given by $q \circ \gamma \circ p$, then γ_* induces a morphism of Hodge structures

$$f_*: p_* H^{i+2m}(X)(m) \rightarrow q_* H^{i+2n}(Y)(n).$$

These rules yield a functor H^i from M_A into the category pure polarizable Hodge structures weight i . The functor $X \mapsto H(X) = \bigoplus H^i(X)$ gives faithful additive embeddings of M_{hom} and M_A into the *PHS* (the faithfulness can be checked using Manin's identity principle [33,42]).

Since M_A is semisimple Abelian, the additivity forces H and H^i to be exact on it as well. H also preserves tensor products and duals. These Hodge structures are not compatible with ungraded correspondences. However, after adjusting weights and summing, the Hodge structures

$$\tilde{H}^{even}(X, p, m) = \bigoplus_j p_* H^{2j+m}(X)(j+m),$$

$$\tilde{H}^{odd}(X, p, m) = \bigoplus_j p_* H^{2j+m+1}(X)(j+m)$$

will give functors from $M_A^u \rightarrow PHS$. Furthermore, $X \mapsto \tilde{H}(X) = \tilde{H}^{even}(X) \oplus \tilde{H}^{odd}(X)$ gives a faithful embedding. When k is arbitrary, similar remarks apply with H replaced by ℓ -adic cohomology.

For any admissible class \mathcal{V} , we can identify $M_A(\mathcal{V})$ with a subcategory of M_A . This need not be a full embedding, since the notion of motivated cycles modeled on \mathcal{V} may be more restrictive than motivated cycles modeled on all of $SPVar_k$. Let $M_A(\mathcal{V})^{full} \subseteq M_A$ be the full subcategory generated by $M_A(\mathcal{V})$. We say that a smooth projective variety Y is *motivated* by \mathcal{V} (or a smooth projective variety X) if $[Y]$ lies in $M_A(\mathcal{V})$ (or $M_A(X)$). More precisely, this means that $[Y]$ is isomorphic in M_A to an object of $M_A(\mathcal{V})$. Replacing $M_A(\mathcal{V})$ by $M_A(\mathcal{V})^{full}$ leads to the more flexible (although harder to control) notion of weak motivation. For example, André [2] has shown that any K3 surface is weakly motivated by an Abelian variety. The corresponding result for motivation is unknown except in special cases such as for Kummer surfaces. X and Y will be called (weakly) *co-motivated* if they are (weakly) motivated by each other.

Lemma 1.1. *Given X, Y in $SPVar_k$, Y is (weakly) motivated by X if and only if there exists a morphism*

$$f: \bigoplus_{m,n} [X]^{\otimes n}(m) \rightarrow [Y]$$

in $M_A(X)$ ($M_A(X)^{full}$) inducing a surjection on cohomology.

Proof. If $[Y]$ lies in $M_A(X)$, then it is a direct summand of some $\bigoplus [X]^n(m)$. Therefore projection yields the desired morphism f .

Conversely, given a morphism f as above. Since $H: M_A \rightarrow PHS$ is faithful and exact, it follows that f is an epimorphism. Therefore $[Y]$ is a summand of $\bigoplus [X]^n(m)$, since M_A is semisimple. \square

Corollary 1.2. *Y is motivated by X if there exists a surjective morphism of varieties $f: X^n \rightarrow Y$.*

Proof. By taking general hyperplane sections, we can find a smooth $g: Z \hookrightarrow X^n$ such that $h = f \circ g$ is surjective, and $\dim Z = \dim Y$. The map $h_*: H(Z) \rightarrow H(Y)$ is surjective since $\frac{1}{\deg h} h^*$ splits it. There f_* is also surjective. \square

Lemma 1.3. *If X is smooth projective variety, its Albanese $Alb(X)$ is motivated by X . If X is a smooth projective curve, X and its Jacobian $J(X)$ are co-motivated.*

Proof. Let $\alpha: X \rightarrow \text{Alb}(X)$ be the Abel–Jacobi map. Since $\text{Alb}(X)$ is generated as a semigroup by the image of α , the map $X^n \rightarrow \text{Alb}(X)$ given by $(x_1, \dots, x_n) \mapsto \alpha(x_1) + \dots + \alpha(x_n)$ is surjective for some n . This proves the first statement.

Suppose that X is a curve. We have just seen that $J(X) = \text{Alb}(X)$ is motivated by X . Since α^* induces a surjection on cohomology, X is also motivated by $J(X)$. \square

Lemma 1.4. *Suppose that X and Y are smooth projective varieties such that there exists a finite collection of motivated correspondences on $X \times Y$ modeled on $\langle X \rangle$ (respectively SPVar_k) whose Künneth components, along Y , generate the cohomology ring $H(Y)$. Then Y is motivated (respectively weakly motivated) by X .*

Proof. Let $d = \dim X$, and let $c_{i,j} \in A_{\text{mot}}^{d+i}(X \times Y)$ denote the classes of the given correspondences. These induce morphisms $[X](-i) \rightarrow [Y]$ in M_A . Products $c_{i_1,j_1} \otimes \dots \otimes c_{i_n,j_n}$ induce morphisms

$$[X]^{\otimes n}(-i_1 - i_2 - \dots - i_n) \rightarrow [Y]^n \xrightarrow{\Delta^*} Y.$$

By assumption, a finite sum of these morphisms yield a map

$$f: \bigoplus_{(i_1, \dots, i_n)} [X]^n(*) \rightarrow [Y]$$

which induces a surjection on cohomology. Therefore we are done by Lemma 1.1. \square

Lemma 1.5. *Let X, Y be smooth projective varieties. Suppose that the diagonal $\Delta \in H(Y \times Y)$ is contained in the algebra generated by products $\mu \times \mu'$, where $\mu, \mu' \in H(Y)$ are Künneth components of motivated correspondences on $X \times Y$ modeled on $\langle X \rangle$ (respectively SPVar_k). Then X motivates (respectively weakly motivates) Y .*

Proof. By assumption, we can write $\Delta = \sum n_{v\xi} v \times \xi$, where v, ξ are products of Künneth components of correspondences on $X \times Y$. Given $\alpha \in H(Y)$, we have

$$\alpha = \Delta_*(\alpha) = \sum n_{v\xi} \langle v, \alpha \rangle \xi,$$

where $\langle v, \alpha \rangle$ denotes $\int v \cup \alpha$. Thus the hypothesis of Lemma 1.4 is fulfilled. \square

Lemma 1.6. *If $H^*(X)$ is generated as an algebra by elements of degree at most d , then X is weakly motivated by a variety of dimension less than or equal to d .*

Proof. If $\dim X \leq d$ there is nothing to prove. Otherwise, let $\iota': Z \hookrightarrow X$ be an intersection of X with $\dim X - d$ hyperplanes in general position. We get an induced morphism $\iota: [X] \rightarrow [Z]$ of motives. Since M_A is semisimple, there exists a morphism $\sigma: [Z] \rightarrow [X]$ satisfying $\iota\sigma\iota = \iota$ (this can be obtained as a composition of splittings $[Z] \rightarrow \text{im}(\iota) \rightarrow [X]$). By the weak Lefschetz theorem, $\iota_*: H^a(X) \rightarrow H^a(Z)$ is injective for $a \leq d$. Therefore the map $H^a(Z) \rightarrow H^a(X)$ induced by σ is surjective when $a \leq d$. By assumption, $H(X)$ is generated, as an algebra, by the elements in the images of these maps. Therefore we are done by Lemma 1.4. \square

We can view the smallest m , for which X is weakly motivated by an m -fold, as a measure of the complexity of X . For such an m , the Hodge structures $H^i(X)$ would have to lie in the tensor category generated by Hodge structures of level at most m . Following Schoen [41], we can find a Hodge theoretic obstruction to this. Given a Hodge structure H , let $\mu(H)$ denote the level of the induced Hodge structure on the Mumford–Tate Lie algebra of H . (Recall that the level of a Hodge structure G is $\max |p - q|$ such that $G^{p,q} \neq 0$.) We have that $\mu(H)$ is bounded above by the twice the level of H , and that

$$\mu(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \leq \max \{ \mu(H_i) \}$$

(see [41, pp. 546–548]). From this it follows that if H lies in the tensor category generated by Hodge structures of level at most m , then $\mu(H) \leq 2m$. Let $\tau(X)$ be Schoen’s invariant, which is half the maximum of $\mu(H^i)$ as H^i varies over all irreducible Hodge substructures $H^i(X)$ of level i for all i . Then from this discussion, we find:

Lemma 1.7. *If X is weakly motivated by an m -dimensional variety, then $\tau(X) \leq m$.*

Schoen [41] gives examples, such as general hypersurfaces of large degree, where $\tau(X) = \dim X$.

2. Singular or non-projective varieties

It will be convenient to extend the previous ideas to the category $\text{Var}_{\mathbb{C}}$ of all varieties over \mathbb{C} . If X is a proper variety which is a rational homology manifold, then it is as good as smooth for our purposes. In particular, we can attach a homological motive $[X]$ to it as follows. Since the rational cohomology of X satisfies Poincaré duality, we have a Gysin map p_* for any resolution $p: \tilde{X} \rightarrow X$. We take $[X] = (\tilde{X}, p^* p_*, 0)$, which is easily seen to be well defined.

More general varieties give rise to mixed motives, in principle. However we will only need the pure part of this structure. This is analogous to passing from a mixed Hodge structure H to the pure structure $\text{Gr}_{\bullet}^W H$. Let $K^b(A)$ denote the homotopy category of bounded complexes in an additive category A . This has a natural triangulated structure. When A is Abelian, we have functors $h^i: K^b(A) \rightarrow A$ given by taking i th cohomology. The following was obtained by Gillet–Soulé [19] and Guillén–Navarro [23].

Theorem 2.1. *Let $k = \mathbb{C}$, then for each $X \in \text{Ob Var}$, there exists a well-defined complex $W(X) \in \text{Ob } K^b(M_{CH})$ such that:*

- (1) *When X is a smooth projective variety, $W(X) \cong [X]$.*
- (2) *W behaves contravariantly for proper maps.*
- (3) *W behaves covariantly for open immersions.*
- (4) $W(X \times Y) \cong W(X) \otimes W(Y)$.
- (5) *If $U \subset X$ is open, there is a natural distinguished triangle*

$$W(U) \rightarrow W(X) \rightarrow W(X - U) \rightarrow W(U)[1].$$

- (6) $h^j(H^i(W(X))) = \text{Gr}_i^W H_c^{i+j}(X)$, where H_c denotes cohomology with compact support.

There are a few cases in which this complex can be made rather explicit. Given a divisor with normal crossings $D = \bigcup D_i$ on a smooth projective variety X , then $W(X - D)$ can be realized by the complex

$$[X] \rightarrow \bigoplus_i [D_i] \rightarrow \bigoplus_{i,j} [D_i \cap D_j] \rightarrow \dots$$

with simplicial coboundaries [19]. Item (6) is essentially given in [19, p. 147], however it can be seen directly for $X - D$ from the above complex.

If X is a smooth projective variety with a finite group action such that the quotient X/G is a variety, $W(X/G)$ is isomorphic to $e[X]$ in degree 0, where $e = (1/\#G) \sum g \in \mathbb{Q}[G]$ [14]. In fact, it is easy to combine these two cases to see that if G acts on (X, D) , $W((X - D)/G)$ is given by

$$e[X] \rightarrow \bigoplus_i e[D_i] \rightarrow \dots$$

Let $W_A(X)$ be the image of $W(X)$ in $K^b(M_A)$. We write $Gr_j[X]$ for $h^j(W_A(X))$. Under the embedding $H : M_A \rightarrow PHS$,

$$H(Gr_j[X]) = \bigoplus_i Gr_i^W H_c^{i+j}(X).$$

The discussion in the previous paragraph implies that if X is smooth with a smooth compactification \bar{X} , $Gr_0[X]$ is a subobject of $[\bar{X}]$.

Corollary 2.2. *If $U \subset Y$ is open, we have an exact sequence*

$$\dots \rightarrow Gr_j[U] \rightarrow Gr_j[Y] \rightarrow Gr_j[Y - U] \rightarrow \dots$$

in M_A .

We will say that an arbitrary variety Y is (weakly) motivated by \mathcal{V} if $W_A(Y)$ is isomorphic a complex in $K^b(M_A(\mathcal{V}))$ (respectively $K^b(M_A(\mathcal{V})^{full})$). If Y is smooth and projective, these notions are equivalent to the previous definitions since $W_A(Y) \cong [Y]$.

Corollary 2.3. *If $U \subset Y$ is open and any two of $U, Y, Y - U$ are (weakly) motivated by \mathcal{V} , then so is the third. If X and Z are (weakly) motivated by \mathcal{V} , then so is $X \times Z$.*

Proof. Since any vertex of a distinguished triangle can be constructed from the other two in terms of mapping cones, the first statement follows. The second statement is evident from the theorem. \square

By induction, we get:

Corollary 2.4. *If Y is a smooth projective variety which can be expressed as a disjoint union $Y = \bigcup Y_i$ of locally closed varieties, such that Y_i is (weakly) motivated by \mathcal{V} . Then so is Y .*

Let us say that a morphism $Y \rightarrow S$ is *cellular* if it is flat and admits a decomposition $Y = \bigcup Y_i$ with Y_i isomorphic to an affine space fibration $\mathbb{A}_S^{n_i}$.

Lemma 2.5. *If $Y \rightarrow S$ is cellular, then Y is motivated by S .*

Proof. This follows from the previous two corollaries. Alternatively, it can be deduced from the isomorphism of graded Chow motives

$$[Y] \cong \bigoplus [S](i),$$

given in [42, 2.6]. \square

Combining this with the previous results gives

Corollary 2.6. *Suppose that Y is a disjoint union $Y = \bigcup Y_i$ of subvarieties admitting cellular maps $Y_i \rightarrow Z_i$ with Z_i (weakly) motivated by \mathcal{V} . Then Y is (weakly) motivated by \mathcal{V} .*

Corollary 2.7. *The blow-up of a smooth projective variety Y along a smooth center V is motivated by the disjoint union $Y \sqcup V$.*

This is an immediate consequence of the previous corollary. It can also be deduced from the blow up sequence [33, Section 9], and this works in any characteristic.

Corollary 2.8. *A uniruled n -dimensional variety is motivated by an $(n - 1)$ -dimensional variety. A unirational n -dimensional variety is motivated by a variety of dimension less than $n - 1$.*

Proof. If X is a smooth uniruled n -fold, then there is a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ with $\dim Y = n - 1$. By resolution of singularities, we can find a sequence of blow-ups $B_N \rightarrow \cdots \rightarrow B_1 \rightarrow \mathbb{P}^1 \times Y$ along smooth centers and a surjective morphism $B_N \rightarrow X$. Then X is motivated by $Y \sqcup C_1 \sqcup \cdots \sqcup C_N$, where C_i are centers of the blow-ups. This has dimension $n - 1$, since the centers have dimension at most $n - 2$. A unirational variety is dominated by an iterated blow-up of \mathbb{P}^n . So it is motivated by the union of the centers. \square

Corollary 2.9. *If Y is a smooth projective variety with a \mathbb{C}^* -action, then Y is motivated by the fixed point set $Y^{\mathbb{C}^*}$.*

Proof. By the Bialynicki-Birula decomposition [8], we can decompose Y into a union $Y = \bigcup Y_i$, where Y_i is a affine space bundle over a component of the fixed point set. \square

From the discussion following Theorem 2.1, and the G -equivariant form of resolution of singularities, we get

Lemma 2.10. *Suppose that the action of a finite group G on a smooth variety extends to a compactification, and that the quotient X/G exists in $\text{Var}_{\mathbb{C}}$. Then X/G is (weakly) motivated by \mathcal{V} if X is.*

The quotient X/G always exists as an algebraic space. Thus we could drop the above requirement by extending the above notions to the category of algebraic spaces. However, we will not need this.

3. The coniveau filtration

Let $FPHS$ be the category of filtered polarizable Hodge structures. This is additive, but not Abelian. Given objects (H, L) and (G, L) , we have Tate twists:

$$L^p(H(c)) = [L^{p+c}H](c)$$

and tensor products:

$$L^p(H \otimes G) = \sum_{i+j=p} L^i H \otimes L^j G.$$

We define the *level filtration* \mathcal{L}^\bullet on a pure Hodge structure H to be the largest Hodge substructure of H satisfying $\mathcal{L}^p \subseteq F^p$. If H is pure of weight m , it follows that \mathcal{L}^p is the maximal substructure with level at most $|m - 2p|$.

Lemma 3.1. *The operation $V \mapsto \mathcal{L}^p V$ gives rise to an exact endofunctor on the category of polarizable Hodge structures. The functor $V \mapsto (V, \mathcal{L}^\bullet V)$ from $PHS \rightarrow FPHS$ is compatible with Tate twists and products.*

Proof. The operation $H \mapsto \mathcal{L}^p H$ is easily seen to be an additive functor. In particular, it preserves direct sums. Since PHS is semisimple, this forces exactness. The remaining properties are straightforward. \square

Let X be a smooth projective variety, the *coniveau filtration* is given by

$$\begin{aligned} N^p H^i(X) &= \sum_{\text{codim } Y \geq p} \ker[H^i(X) \rightarrow H^i(X - Y)] \\ &= \sum_{\text{codim } Y = q \geq p} \text{im}[H^{i-2q}(\tilde{Y})(-q) \rightarrow H^i(X)] \end{aligned}$$

where Y ranges over closed subvarieties; in the second expression $\tilde{Y} \rightarrow Y$ are chosen desingularizations. Since the level of $H^{i-2q}(\tilde{Y})(-q)$ is bounded by $i - 2p$, we have an inclusion

$$N^p H^i(X) \subseteq \mathcal{L}^p H^i(X).$$

The generalized Hodge conjecture asserts that equality holds. This would imply functoriality of the coniveau filtration. Fortunately, this can be checked directly. The following is proven in [5]:

Theorem 3.2. *The coniveau filtration N^\bullet is preserved by pushforwards, pullbacks, and products. More precisely:*

- (1) If $f: X \rightarrow Y$ is a map of smooth projective varieties of dimensions n and m respectively, then

$$f_*(N^p H^i(X)) \subseteq N^p(H^{i+2(m-n)}(Y)(m-n)).$$

- (2) If f is as above, then

$$f^*(N^p H^i(Y)) \subseteq N^p H^i(X).$$

- (3) $N^p I(H^i(X)) \otimes N^q(H^j(Y)) \subseteq N^{p+q} H^{i+j}(X \times Y)$.

Corollary 3.3. *The action of a correspondence preserves the coniveau filtration.*

This allows us to define the coniveau filtration of a motive by

$$N^j H^i(X, p, m) = p_* N^j(H^{i+2m}(X)(m)).$$

4. The conjectures

We work over \mathbb{C} . We recall the basic conjectures as conditions on a fixed smooth projective variety X .

$D(X)$: Homological equivalence coincides with numerical equivalence on X .

$B(X)$: For each $i \leq \dim X$, there exists an algebraic correspondence inducing an isomorphism

$$v^i: H^{\dim X+i}(X, \mathbb{Q}) \xrightarrow{\sim} H^{\dim X-i}(X, \mathbb{Q}).$$

$HC(X)$: Any Hodge (i.e. rational (p, p)) cycle on X is algebraic.

$GHC(X)$: $N^p H^i(X) = \mathcal{L}^p H^i(X)$ for all i, p .

$AC(X)$: All Hodge cycles on X are motivated in the widest sense (i.e. motivated with respect to $SPVar_{\mathbb{C}}$).

D and B are among Grothendieck's standard conjectures [22,30,31]. B is called the Lefschetz standard conjecture. HC and GHC are the Hodge and generalized Hodge conjectures respectively. AC is due to André; it sits in between the Hodge conjecture and Deligne's conjecture [17] on the absoluteness of Hodge cycles. The Hodge conjecture is well known to be equivalent to the fullness of the embedding $M_{hom} \rightarrow PHS$. A similar interpretation holds for AC in terms of $M_A \rightarrow PHS$. We have implications $GHC(X) \Rightarrow HC(X) \Rightarrow D(X)$ and $D(X \times X) \Leftrightarrow B(X)$ [30,31]. It is straightforward to extend some of these conjectures to motives. Given M in M_A , $GHC(M)$ (respectively $HC(M)$) would assert $N^p H^i(M) = \mathcal{L}^p H^i(M)$ for all indices (respectively for $i = 2p$). The formulations of HC by Jannsen [28, 7.9] and GHC by Lewis [32, Appendix A] for a general variety X are equivalent to $HC(Gr_0[X])$ and $GHC(Gr_0[X])$ respectively. We should emphasize that while technically convenient, these extensions to motives and general varieties are no stronger than the original conjectures.

The following is a repackaging of results of André, Grothendieck, Jannsen and Kleiman.

Theorem 4.1. *The following are equivalent:*

- (1) $M_{\text{hom}}(X)$ is semisimple and Abelian.
- (2) $M_{\text{hom}}(X) \rightarrow M_A(X)$ is an equivalence.
- (3) Numerical equivalence coincides with homological equivalence on X and all its powers.
- (4) The Lefschetz standard conjecture holds for X .

Proof. The equivalence of (3) and (4) is proven in [31, Proposition 5.1]. (2) implies (1) by [2, 4.2]. (1) implies (3) by the first step of the proof of [29, Theorem 1]. Finally assume (4). Then conjecture B holds for all powers X^N [31, Proposition 4.2]. A motivated cycle modeled on the category generated by X , is an expression of the form

$$\gamma = p_{X^n*}(\alpha \cup * \beta)$$

where $\alpha, \beta \in A(X^{n+m})$, and $*$ is the Lefschetz involution [2]. Since $B(X^{n+m})$ holds, $*\beta$ would be algebraic by [2, Proposition 1.2] and [31]. Therefore γ would be algebraic. Thus (2) holds. \square

We now come to the main point.

Lemma 4.2. *Suppose that X and Y are smooth projective varieties such that Y is motivated by X . If X and all its powers satisfies one of the conjectures (D , B , HC , GHC) stated above, then the same conjecture holds for Y and all its powers. If Y is weakly motivated by X and AC holds for all powers of X , then it holds for all powers of Y .*

Proof. Since Y^n is also motivated by X , it suffices to prove the conjectures hold for Y alone.

Suppose that $D(X^n)$ holds for all n . Then the motive $[Y] \in M_A(X)$ is a direct summand of some $\mathcal{E} = \bigoplus X^{n_i}(j_i)$ with complement say Y' . Given $\gamma \in H(\mathcal{E})$, let us write γ_1 and γ_2 for its component with respect to the decomposition $H(\mathcal{E}) = H(Y) \oplus H(Y')$. Since $M_A(X)$ is equivalent to $M_{\text{hom}}(X)$ by the previous theorem, this decomposition of \mathcal{E} lies in $M_{\text{hom}}(X)$, therefore γ_i are both algebraic if and only if γ is. Suppose that $\alpha \in H(Y)$ is an algebraic cycle which is numerically equivalent to 0. We can lift it to a class $\beta \in H(\mathcal{E})$ with $\beta_1 = \alpha$ and $\beta_2 = 0$. For any other algebraic cycle γ , we have $\gamma \cdot \beta = \gamma_1 \cdot \alpha = 0$. Therefore β is numerically trivial, and consequently homologically trivial.

Since the statements $\forall n, D(X^n)$ and $\forall n, B(X^n)$ are equivalent, case B follows from the previous one.

Suppose that $HC(X^n)$ or $GHC(X^n)$ holds for all n . We can repeat the previous argument to write $H(\mathcal{E}) = H(Y) \oplus H(Y')$. Any Hodge cycle $\alpha \in H(Y)$ can be lifted to a Hodge cycle on β on \mathcal{E} with $\beta_1 = \alpha$ and $\beta_2 = 0$. Assuming $HC(X^n)$, β would have to be algebraic, and therefore α is also algebraic. Assuming $GHC(X^n)$, the equality $N^p H(\mathcal{E}) = \mathcal{L}^p H(\mathcal{E})$ forces a similar equality for Y .

Finally, suppose that $AC(X^n)$ holds for all n and that Y is weakly motivated by X . The argument of the previous paragraph with “algebraic” replaced by “motivated” and within $M_A(X)^{\text{full}}$ shows $AC(Y)$. \square

For conjecture B , see [13, Theorem 5.11] for a refinement.

Corollary 4.3. *The Lefschetz standard conjecture holds for any variety motivated by a curve or surface. In particular, it holds for a uniruled threefold, a unirational fourfold.*

Proof. The Lefschetz conjecture for a curve or surface follows from the Lefschetz (1, 1) theorem. Therefore it holds for a power of a curve or a surface by [31, Proposition 4.3.1]. The second statement follows from Corollary 2.8. \square

We can recover a result of Lieberman that the Lefschetz conjecture holds for an Abelian variety, since its cohomology is generated by H^1 . We also note that “most” varieties are not motivated by surfaces by Lemma 1.7.

Corollary 4.4. *If X is weakly motivated by an Abelian variety, then AC holds for X and all its powers.*

Proof. This follows from [2, Theorem 0.6.2]. \square

We can see that the hypothesis holds for a unirational threefold by Corollary 2.8, or a smooth projective variety X whose cohomology is generated as an algebra by $H^1(X)$ by Lemma 1.6. Additional examples, provided by [2,3], include K3 surfaces and cubic hypersurfaces of dimension at most 6.

5. Fourier–Mukai transforms

We return to the case of a general field k . As we saw earlier, in order to prove that a smooth projective variety Y is motivated by another such variety X , it is necessary to find a suitable correspondence from a sum of powers of X to Y . When Y is a moduli space of objects on X , the correspondence can often be constructed with the help of a Fourier–Mukai transform or something close to it. Fix a sheaf E on $X \times Y$ or more generally an object in the bounded derived category of coherent sheaves $D(X \times Y)$. The Fourier–Mukai transform with kernel E is the exact (i.e. triangle preserving) functor $\Phi_E : D(X) \rightarrow D(Y)$ given by

$$\Phi_E(\mathcal{F}) = \mathbb{R}p_{Y*}(p_X^*\mathcal{F} \otimes E)$$

where p_X, p_Y denote the projections. Given $F \in D(Y \times Z)$, the composition $\Phi_F \circ \Phi_E$ is again a Fourier–Mukai transform:

$$F \circ E = \mathbb{R}p_{XZ*}(p_{XY}^*E \otimes p_{YZ}^*F).$$

Furthermore, the functor Φ_E has left and right adjoints which can also be realized as Fourier–Mukai transforms. Specifically, if $E^T \in D(Y \times X)$ is the “transpose” of E and $E^{T*} = \mathbb{R}\mathcal{H}om(E, \mathcal{O}_{Y \times X})$ its dual, then the right adjoint is $\Phi_{E^{T*} \otimes \omega_X[\dim X]}$. Proofs of these facts can be found in [37,39].

Given an object E in $D(X \times Y)$, we can pass to a K_0 -correspondence $\chi(E) = \sum (-1)^i h^i(E) \in K_0(X \times Y)$. The Künneth formula implies $\chi(F \circ E) = \chi(F) \circ \chi(E)$. Next, we construct a functor, which we call the Mukai functor $\mu : M_{K_0} \rightarrow M_{CH}^u$. It is enough to describe this on Cor_{K_0} . The putative functor μ sends X in M_{K_0} to $[X]$. Given $e \in K_0(X \times Y)$, define $\mu(e) = ch(e) \cdot \sqrt{td(X \times Y)}$, where ch is the Chern character

$$ch : K_0(\cdot) \rightarrow CH(\cdot) \otimes \mathbb{Q},$$

and

$$\sqrt{td()} = 1 + \frac{c_1()} {4} + \frac{c_1()^2} {96} + \frac{c_2()} {24} + \dots$$

is the formal square root of the Todd class of the tangent bundle.

Lemma 5.1. μ is a functor.

Proof. Let $\delta: X \rightarrow X \times X$ be the diagonal embedding, and $\Delta = im(\delta)$. The classes $O_\Delta \in K_0(X \times X)$ and $[\Delta] \in H^*(X \times X)$ represents the identity in their respective categories. From standard properties [18, Example 3.2.4],

$$\delta^*td(X \times X) = td(X)^2.$$

Applying the Grothendieck–Riemann–Roch theorem [18, Theorem 15.2] yields

$$\begin{aligned} ch(\delta_*O_X)td(X \times X) &= \delta_*(ch(O_X)td(X)) \\ &= \delta_*(ch(O_X)\delta^*\sqrt{td(X \times X)}) \\ &= \delta_*(ch(O_X))\sqrt{td(X \times X)} \\ &= [\Delta]\sqrt{td(X \times X)}. \end{aligned}$$

Thus $\mu(O_\Delta) = [\Delta]$ as required.

Given $e \in K_0(X \times Y)$ and $g \in K_0(Y \times Z)$, a second application of Grothendieck–Riemann–Roch gives:

$$\begin{aligned} \mu(g \circ e) &= ch(p_{XZ*}(p_{XY}^*e \cdot p_{YZ}^*g))\sqrt{td(X \times Z)} \\ &= p_{XZ*}(ch(p_{XY}^*e \cdot p_{YZ}^*g) \cdot p_X^*\sqrt{td(X)}p_Y^*td(Y)p_Z^*\sqrt{td(Z)}) \\ &= p_{XZ*}(p_{XY}^*[ch(e) \cdot \sqrt{td(X \times Y)}] \cdot p_{YZ}^*[ch(g) \cdot \sqrt{td(Y \times Z)}]) \\ &= \mu(g) \circ \mu(e). \quad \square \end{aligned}$$

The functor μ is easily seen to be additive. However, it is not compatible with the tensor structures. A similar argument involving Grothendieck–Riemann–Roch yields the following less precise result.

Lemma 5.2. Given $e \in K_0(X \times Y)$ and $g \in K_0(Y \times Z)$, the Chern classes of $g \circ e$ lie in the algebra generated by $\{\epsilon_{i,a} \times \gamma_{j,b}\}$ where

$$c_i(e) = \sum \epsilon_{ia} \times \epsilon'_{ia}, \quad c_i(g) = \sum \gamma'_{ia} \times \gamma_{ia}$$

are the Künneth decompositions of the above Chern classes.

Proposition 5.3. *Suppose that X and Y are smooth projective varieties with $E \in D(Y \times X)$ an object such that $\Phi_E : D(Y) \rightarrow D(X)$ is fully faithful. Then there is a split epimorphism of graded Chow motives*

$$\bigoplus [X](i)^{\oplus n_i} \rightarrow [Y].$$

In particular, Y is motivated by X .

For the proof we need

Lemma 5.4. *Suppose that $F : A \rightarrow B$ is a fully faithful functor with a right adjoint $G : B \rightarrow A$. Then $G \circ F$ is naturally equivalent to the identity on A .*

Proof. We have

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(F(M), F(N)) \cong \mathrm{Hom}(M, G \circ F(N)).$$

Thus $N \cong G \circ F(N)$ since they represent the same functor. \square

Proof of Proposition 5.3. By the results stated earlier, Φ_E has a right adjoint of the form Φ_F with $F \in D(Y \times E)$. The previous lemma shows that this is a left inverse. Therefore $\mu(f \circ e) = \mathrm{id}_Y$, where $e = \chi(E)$ and $f = \chi(F)$. Thus $\mu(f) : [X] \rightarrow [Y]$ gives a split epimorphism in M_{CH}^u . After decomposing $\mu(f)$ into its homogeneous components, we get a surjection $\bigoplus [X](i)^{n_i} \rightarrow [Y]$ in M_{CH} . \square

Corollary 5.5. *Suppose that X and Y are smooth projective varieties with $E \in D(Y \times X)$ an object such that:*

- (1) $\mathrm{Ext}^i(E_s, E_t) = 0$ for all i when $s \neq t$ (where $E_t = E|_{\{t\} \times X}$),
- (2) $\mathrm{Hom}(E_t, E_t) = k$,
- (3) $\mathrm{Ext}^i(E_t, E_t) = 0$ for all $i > \dim Y$.

Then there is a split epimorphism of graded Chow motives

$$\bigoplus [X](i)^{\oplus n_i} \rightarrow [Y].$$

In particular, Y is motivated by X .

Proof. Under the above conditions Φ_E is fully faithful by a theorem of Bondal and Orlov [11, Theorem 3.3]. \square

The hypothesis of the next corollary may seem strange at first glance, however natural examples of pairs of varieties with equivalent derived categories exist [11,37,39].

Corollary 5.6. *Suppose that X and Y are smooth projective varieties whose derived categories are equivalent as triangulated categories. Then the ungraded Chow motives of X and Y are isomorphic. Consequently, X and Y are co-motivated.*

Proof. We appeal to a theorem of Orlov [39, Theorem 3.2.1] which shows that the equivalence $D(X) \rightarrow D(Y)$ and its inverse would be induced by Fourier–Mukai transforms. \square

The hypothesis of Corollary 5.5 requires that $\text{Ext}^\bullet(E_s, E_t)$ is supported on the diagonal. Unfortunately, this is rather restrictive. The following alternative form will be applied later on.

Theorem 5.7. *Let Y and X be smooth projective varieties over a field k . Let $E \in D(Y \times X)$ be an object such that:*

- (1) $\text{Hom}(E_s, E_t) = 0$ if $s \neq t$ and k otherwise.
- (2) $\dim \text{Ext}^1(E_t, E_t) = \dim Y$.
- (3) $\text{Ext}^i(E_s, E_t) = 0$ for $i > 1$.

Then Y is motivated by X .

The following proposition occurs implicitly in [7].

Proposition 5.8. *Let Y, X , and E satisfy above conditions. Then $[\Delta] = c_{\dim Y}(E^* \circ E^T)$ in $CH^*(Y \times Y)$.*

Proof. The arguments given in [7] carry over with very little modification. We set

$$F = \mathbb{R}p_{YY*}\mathbb{R}\text{Hom}(p_{YX}^*E, p_{XY}^*E^T) \cong E \circ E^T.$$

By our assumptions, F as above can be represented by a complex of vector bundles $f: F^0 \rightarrow F^1$. For any $(s, t) \in Y \times Y$, we have

$$0 \rightarrow \text{Hom}(E_s, E_t) \rightarrow F_{s,t}^0 \xrightarrow{f(s,t)} F_{s,t}^1 \rightarrow \text{Ext}^1(E_s, E_t) \rightarrow 0.$$

The Hom above is supported on the diagonal Δ . Thus Δ can be identified with the degeneracy locus of the map f . We note that by our assumptions, the codimension of Δ is $\dim \text{Ext}^1(E_t, E_t) = \text{rank } F^1 - \text{rank } F^0 + 1$. This is the expected codimension, therefore we are in a position to compute the class $[\Delta]$ by Porteous' formula [18, Theorem 14.4], to obtain formula of the proposition. \square

Proof of Theorem 5.7. This is an immediate consequence of the last proposition, Lemmas 1.5 and 5.2. \square

6. GHC for general Jacobians

We make a short digression to prove the generalized Hodge conjecture for powers of a general curve. The result may be known to experts, but we give the proof for lack of a suitable reference. Given a complex Abelian variety X , let $\text{Hdg}(X)$ denote the Hodge (or special Mumford–Tate) group of $H = H^1(X)$. This is the smallest \mathbb{Q} -algebraic subgroup of $GL(H)$ whose real points contain the image of the action $U(1) \rightarrow GL(H \otimes \mathbb{R})$ induced by the Hodge structure. Given a polarization ψ of X , the Lefschetz group $\text{Lef}(X)$, is the centralizer of $\text{End}(X) \otimes \mathbb{Q}$ in

$Sp(H^1(X), \psi)$. The Lefschetz group turns out to be independent of the polarization, and it always contains the Hodge group. The significance of these groups stems from the fact that the invariants of $H^*(X^n)$ under $Hdg(X)$ (respectively $Lef(X)$) are precisely the Hodge classes (respectively sums of products of divisor classes). In particular, $HC(X^n)$ holds for all n whenever these groups coincide. Further discussion along with references can be found in [20,38].

The characterization of Mumford–Tate groups [17, p. 43] together with [16, 7.5] (see also [41, 2.2–2.3]) yields:

Lemma 6.1. *Given a polarized integral variation of Hodge structure V over a smooth irreducible complex variety T , there exists a countable union of proper analytic subvarieties $S \subset T$ such that $Hdg(V_t)$ contains a finite index subgroup of the monodromy group*

$$\text{image}[\pi_1(S, t) \rightarrow GL(V_t)]$$

for $t \notin S$.

Theorem 6.2. (Hazama [26]) *Let X be an Abelian variety satisfying $Hdg(X) = Lef(X)$ and such that all simple factors are of types I or II in Albert's classification, then the generalized Hodge conjecture holds for X .*

Corollary 6.3. *If X is as above, then the generalized Hodge conjecture holds for all powers of X .*

Proof. It can be checked that $Hdg(X^k) = Hdg(X)$. (This is obvious from the Tannakian viewpoint, since $H^1(X)$ and $H^1(X^k) = H^1(X)^k$ generate the same tensor category.) Also $Lef(X) = Lef(X^k)$ [35, Corollary 4.7]. Therefore X^k satisfies the same conditions as the theorem. \square

Corollary 6.4. *If $E = \text{End}(X) \otimes \mathbb{Q}$ is a totally real number field such that $\dim X/[E : \mathbb{Q}]$ is odd then the generalized Hodge conjecture holds for all powers X .*

Proof. The conditions imply that X is simple of type I. The equality $Hdg(X) = Lef(X)$ follows from [40, Theorem 1]. \square

Proposition 6.5. *There exists a countable union S of proper Zariski closed sets in the moduli space $M_g(\mathbb{C})$ of curves of genus $g \geq 2$, such that if $X \in M_g(\mathbb{C}) - S$ then the generalized Hodge conjecture holds for all powers of its Jacobian $J(X)$.*

We shall call such a curve *very general*.

Proof. Choose $n \geq 3$ and let $M_{g,n}$ be the fine moduli space of smooth projective curves of genus g with level n structure [1, 13.4]. Let $\pi : \mathcal{X} \rightarrow M_{g,n}$ be the universal curve. Lemma 6.1 applied to $\mathbb{R}^1\pi_*\mathbb{Z}$ shows that there exist a countable union of proper subvarieties $S' \subset M_{g,n}(\mathbb{C})$ such that a finite index subgroup of the monodromy group

$$\Gamma = \text{image}[\pi_1(M_{g,n}, t) \rightarrow GL(H^1(\mathcal{X}_t))]$$

is contained in $Hdg(\mathcal{X}_t)$ for each $t \notin S'$. Let S be the image of S' in $M_g(\mathbb{C})$. By Teichmüller theory, any finite index subgroup of Γ is seen to be Zariski dense in the symplectic group (see

[24, Section 12]). Hence the Hodge group contains the symplectic group whenever $t \notin S$. But this forces

$$\mathrm{Hdg}(J(\mathcal{X}_t)) = \mathrm{Lef}(J(\mathcal{X}_t)) = \mathrm{Sp}(H^1(\mathcal{X}_t)).$$

Fix $X = \mathcal{X}_t$, with t as above. We will show that $\mathrm{End}(J(X)) \otimes \mathbb{Q} = \mathbb{Q}$, and this will finish the proof by Corollary 6.4. The natural map

$$\mathrm{End}(J(X)) \otimes \mathbb{Q} \rightarrow \mathrm{End}(H^1(X))$$

is injective, and the image lies in the ring $\mathrm{End}_{\mathrm{HS}}(H^1(X))$ of endomorphisms of the Hodge structure $H^1(X)$. This is contained in the space of $\mathrm{Hdg}(X)$ -equivariant endomorphisms of $H^1(X)$. Since $\mathrm{Hdg}(X)$ is the full symplectic group, it acts irreducibly on $H^1(X)$. Therefore Schur's lemma implies that $\mathrm{End}(X) \otimes \mathbb{Q} = \mathbb{Q}$ as claimed. \square

7. Application to moduli spaces

Let X be a smooth projective curve defined over \mathbb{C} . Then the moduli space of stable vector bundles of coprime rank n and degree d over X is a smooth projective fine moduli space [43]. More generally, we can consider the moduli space M of stable parabolic bundles with respect to a given collection of weights [loc. cit.]. Under appropriate numerical conditions on n, d and the weights [9, Section 2], which we assume, M is again a smooth projective fine moduli space.

Theorem 7.1. *With X and M as above, M is motivated by $J(X)$.*

The special case where M is moduli space of vector bundles was due to del Baño. We give a separate proof for this case which is entirely self contained.

Proof for vector bundles. Since M is fine, there is a Poincaré bundle E on $M \times X$. This satisfies the hypothesis of Theorem 5.7, therefore M is motivated by X , and hence by $J(X)$ by Lemma 1.3. \square

Proof for parabolic bundles. Biswas and Raghavendra [10] have shown that $H(M)$ is generated by the Künneth components of Chern classes of certain universal sheaves on $X \times M$. Therefore we can apply Lemma 1.4. \square

The first part of the following is due to Biswas and Narasimhan [9].

Corollary 7.2. *M (as above) satisfies the Lefschetz standard conjecture and AC.*

The following corollaries can be deduced by combining the theorem with known criteria for the validity of Hodge conjecture for Abelian varieties.

Corollary 7.3. *If X is*

- (1) *a curve of genus 2 or 3,*
- (2) *a curve of prime genus such that the Jacobian is simple, or*

- (3) a Fermat curve $x^m + y^m + z^m = 0$ with m prime or less than 21, or
 (4) a curve admitting a surjection from a modular curve $X_1(N)$,

then the Hodge conjecture holds for M . If X is a very general curve, then the generalized Hodge conjecture holds for M .

Proof. A detailed explanation of the ideas involved can be found in [4]. In outline, for (3) we can apply a theorem of Shioda [44, Theorem IV]. The remaining results follow from the equality of the Hodge and Lefschetz groups of $J(X)$. For (2), this equality can be obtained from work of Tankeev and Ribet [40, p. 525]. For (1), the equality is due Mumford although unpublished. However, a proof can be found in [36]. In case (4), the equality is given by work of Hazama and Murty [25].

The last statement follows from Proposition 6.5. \square

Let X be a smooth projective surface. Fogarty has shown that the Hilbert scheme M of zero-dimensional subschemes of fixed length n is smooth and projective (see [21]).

Theorem 7.4. (de Cataldo–Migliorini) M is motivated by X .

Proof. Let $X^{(n)} = S^n X$ denote the n th symmetric power. let $X^{[n]} = M$ be the Hilbert scheme of zero-dimensional subschemes of X of length n . There are canonical morphisms $p: X^n \rightarrow X^{(n)}$ and $\psi: X^{[n]} \rightarrow X^{(n)}$. The map ψ is birational. These spaces have a natural stratification. Given a partition $\lambda = (n_1, n_2, \dots, n_k)$ of n (i.e. a non-strictly decreasing sequence of positive integers summing to n), let

$$X_\lambda^{(n)} = \{p(x_1, \dots, x_n) \mid x_1 = x_2 = \dots = x_{n_1} \neq x_{n_1+1} = \dots = x_{n_1+n_2} \neq \dots\}$$

and let

$$X_\lambda^{[n]} = \psi^{-1} X_\lambda^{(n)}.$$

These are locally closed subsets of $X^{(n)}$ and $X^{[n]}$ which will be regarded as subschemes with reduced structure. We will argue that each $X_\lambda^{[n]}$ is motivated by X . Then the theorem will follow by Corollary 2.6.

The scheme $X_{(n)}^{[n]}$ parameterizes zero-dimensional subschemes with support at a single point. There is a morphism $\pi_n: X_{(n)}^{[n]} \rightarrow X$ which sends a subscheme to its support. Let $U_k \subset X^k$ be the open subset of k -tuples with distinct components. For a partition $\lambda = (n_1, \dots, n_k)$ of n , define

$$X_\lambda^{(n)} = U_k \times_{X^k} \prod_{i=1}^k X_{(n_i)}^{[n_i]}.$$

Göttsche [21, 2.1.4, 2.2.4] has shown that π_n is a Zariski locally trivial fiber bundle where the fiber is smooth, projective and has a cellular decomposition. Then Corollary 2.6 implies that $[\prod X_{(n_i)}^{[n_i]}]$ is motivated by X . U_k is motivated by X , since it is the complement of a diagonal in X^k . Therefore $X_\lambda^{(n)}$ is also motivated by X .

Göttsche [21, 2.3.3] has shown that $X_\lambda^{[n]}$ is a quotient of $X_\lambda^{(n)}$ by a subgroup of S_n . It follows that $X_\lambda^{[n]}$ is also motivated by X by Lemma 2.10. \square

Corollary 7.5. *The Lefschetz standard conjecture holds for M .*

Corollary 7.6. *If X is an Abelian surface over \mathbb{C} , the Hodge conjecture holds for M .*

Proof. As noted earlier, HC holds for all powers of X . \square

Corollary 7.7. *Let X be a smooth projective surface over \mathbb{C} with Kodaira dimension $\kappa(X) \leq 0$, then the conjecture AC holds for M .*

Proof. It suffices by the results of [2, Theorems 0.62, 0.63] to prove that X , and therefore M , is motivated by an Abelian variety, a K3 surface or (for trivial reasons) a projective space. Clearly X can be assumed minimal since it is co-motivated with a minimal model for it. Using classification of surfaces [6], we see that X rational, ruled over a curve C , or else there exist a surjective map $S \rightarrow X$ with S Abelian or K3. In the last two cases, X is motivated $J(C)$ or S as required. \square

Let X be an Abelian or K3 surface over \mathbb{C} with an ample line bundle H . Let M be the moduli space of H -stable of rank r torsion free sheaves with fixed Chern classes c_1, c_2 . Mukai has shown that M is always smooth. Under appropriate conditions on the invariants, M is also projective. See [27] for further details.

Theorem 7.8. *Let X and M be as in the previous paragraph with M projective. Then M is motivated by X .*

Proof. By a theorem of Markman [34], $H(M)$ is generated by the Künneth components of Chern classes of a quasi-universal sheaf \mathcal{E} on $X \times M$. Therefore we can apply Lemma 1.4. \square

Corollary 7.9. *If X is Abelian of K3 then $B(M)$ and $AC(M)$ hold, and $HC(M)$ also holds in the Abelian case.*

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